The twisted conjugacy problem for pairs of endomorphisms in nilpotent groups

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October 19, 2009

Abstract

An algorithm is constructed that, when given an explicit presentation of a finitely generated nilpotent group G, decides for any pair of endomorphisms $\varphi, \psi : G \to G$ and any pair of elements $u, v \in G$, whether or not the equation $(x\varphi)u = v(x\psi)$ has a solution $x \in G$. Thus it is shown that the problem of the title is decidable. Also we present an algorithm that produces a finite set of generators of the subgroup (equalizer) $Eq_{\varphi,\psi}(G) \leq G$ of all elements $u \in G$ such that $u\varphi = u\psi$.

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2000 Mathematics Subject Classification. Primary 20F10. Secondary 20F18; 20E45; 20E36.

Keywords. nilpotent groups, conjugacy classes, twisted conjugacy, endomorphism, automorphism.

1 Introduction

Let G be a group, and $u, v \in G$. Given an endomorphism $\varphi \in End(G)$, one says that u and v are $\varphi-$*twisted conjugate*, and one writes $u \sim_\varphi v$, if and only if there exists $x \in G$ such that $u = (x\varphi)^{-1}vx$, or equivalently $(x\varphi)u = vx$. More generally, given a pair of endomorphisms $\varphi, \psi \in End(G)$, one says that

*The first author was partially supported by RFBR grant 07-01-00392.

the elements u, v are $\varphi, \psi-$*twisted conjugate*, and one writes $u \sim_{\varphi,\psi} v$, if and only if there exists an element $x \in G$ such that

$$(x\varphi)u = v(x\psi). \tag{1}$$

The recognition of twisted conjugacy classes with respect to a given pair of endomorphisms $\varphi, \psi \in End(G)$ in the case of any finitely generated nilpotent group G is the main concern of this paper.

2 Preliminary results

It is well known (see, for example, [2], [4]) that most of algorithmic problems for finitely generated nilpotent groups are decidable. In particular, the standard conjugacy problem is decidable in any finitely generated nilpotent group [1]. Some undecidable problems exist too. The endomorphism (see [5]) and the epimorphism (see [3]) problems are among them, as well as impossibility to decide if a given equation has a solution in a given finitely generated nilpotent group (see [5]).

For the purpose of this paper we need in the following result of special interest.

Proposition 2.1. Let G be a finitely generated nilpotent group. Let $\varphi, \psi \in End(G)$ be a pair of endomorphisms of G. Then there is an algorithm which finds a finite set of generators of the subgroup (equalizer)

$$Eq_{\varphi,\psi}(G) = \{x \in G | x\varphi = x\psi\}. \tag{2}$$

Proof. Let G be abelian. Then $Eq_{\varphi,\psi}(G) = ker(\varphi - \psi)$, thus there is a standard procedure to find a generating set for it.

Let G be a finitely generated nilpotent group of class $c+1 \geq 2$. Suppose by induction that there is an algorithm which finds for any finitely generated nilpotent group H of class $\leq c$ and any pair of endomorphisms $\alpha, \beta \in End(H)$ a finite set of generators of the equalizer $Eq_{\alpha,\beta}(H)$.

Let $C = \gamma_c(G)$ be the last non trivial member of the lower central series of G. Then the quotient $H = G/C$ has class c. Since C is invariant for every endomorphism of G we can consider the induced by $\varphi, \psi \in End(G)$ endomorphisms $\bar{\varphi}, \bar{\psi} \in End(H)$. By the assumption we can construct a finite set of generators of the equalizer $Eq_{\bar{\varphi},\bar{\psi}}(H) \leq H$. Let G_1 be the full preimage of $Eq_{\bar{\varphi},\bar{\psi}}(H)$ in G. We call G_1 a $C-$equalizer of φ, ψ, and we write $G_1 = Eq_{C,\varphi,\psi}(G)$. By definition

$$Eq_{C,\varphi,\psi}(G) = \{g \in G | g\varphi = c_g(g\psi), \text{where } c_g \in C\}. \tag{3}$$

Obviously, $Eq_{\varphi,\psi}(G) \leq Eq_{C,\varphi,\psi}(G)$, and $C \leq Eq_{C,\varphi,\psi}(G)$.

Now we define a map

$$\mu : Eq_{C,\varphi,\psi}(G) \to C \text{ by } \mu(g) = c_g. \tag{4}$$

Easily to see that this map μ is a homomorphism, and that the derived subgroup $(Eq_{C,\varphi,\psi}(G))'$ lies in $ker(\mu)$.

We conclude that $Eq_{\varphi,\psi}(G) = ker(\mu)$. So, a generating set for $Eq_{\varphi,\psi}(G)$ can be derived by the standard procedure.

$\square$

3 The twisted conjugacy problem for pairs of endomorphisms

Let G be a finitely generated group, and $\varphi, \psi \in End(G)$ be any pair of endomorphisms. Let $u \sim_{\varphi,\psi} v$ be a pair of $\varphi, \psi-$twisted conjugate elements of G. We write $\{u\}_{\varphi,\psi}$ for the $\varphi, \psi-$twisted conjugacy class of element $u \in G$.

The question about $\varphi, \psi-$twisted conjugacy of given elements $u, v \in G$ can be reduced to the case where one of the elements is trivial. To do this we change φ to $\varphi' = \varphi \circ \sigma_u$, where $\sigma_u \in Aut(G)$ is the inner automorphism $h \mapsto u^{-1}hu$. Hence $(x\varphi)u = v(x\psi)$ if and only if

$$x\varphi' = w(x\psi), \tag{5}$$

where $w = u^{-1}v$.

Now we are ready to prove our main result about twisted conjugacy in finitely generated nilpotent groups.

Theorem 3.1. *Let G be a finitely generated nilpotent group of class $c \geq 1$. Then there exists an algorithm which decides the twisted conjugacy problem for any pair of endomorphisms $\varphi, \psi \in End(G)$.*

Proof. Induction by c. For $c = 1$ (abelian case) the statement is obviously true.

Suppose that the statement is true in the case of any finitely generated nilpotent group N of class $\leq c - 1$.

Let $u, v \in G$ be any pair of elements, and $\varphi, \psi \in End(G)$ be any pair of endomorphisms. We change φ to $\varphi' = \varphi \circ \sigma_u$, and write the equation (5) with $w = u^{-1}v$, as were explained above. Since the last non trivial member C of the lower central series of G is invariant for every endomorphism, we decide the twisted conjugacy problem in G/C with respect to the induced by φ', ψ endomorphisms $\bar{\varphi}, \bar{\psi} \in End(G/C)$ and the induced by w element $\bar{w} \in G/C$.. More exactly, we decide if there exists an element $\bar{x} \in G/C$ for which

$$\bar{x}\bar{\varphi}' = \bar{w}(\bar{x}\bar{\psi}). \tag{6}$$

By our assumption we can decide this problem effectively. If such element $\bar{x}$ does not exist the element x does not exist too.

Suppose that $\bar{x}$ exists. Then there is an element $x_1 \in G$ for which

$$x_1\varphi' = cg(x_1\psi), \tag{7}$$

where $c \in C$. If $x_2\varphi' = c'g(x_2\psi)$ for some element $x_2 \in G$ and $c' \in C$, we derive that

$$(x_2^{-1}x_1)\varphi' = c''((x_2^{-1}x_1)\psi), \tag{8}$$

where $c'' = (c')^{-1}c \in C$. Thus $x_2^{-1}x_1 \in Eq_{C,\varphi',\psi}(G)$. In the case when $x_2 = x$ is a solution of (5) we have $c'' = c$. Conversely, the equality $c'' = c$ means that there is a solution x of (5).

By Proposition 2.1 we construct a finite generating set of $Eq_{\bar{\varphi}',\bar{\psi}}(G/C)$, and so we can construct a finite generating set of its full preimage:

$$Eq_{C,\varphi,\psi}(G) = gp(g_1, ..., g_l). \tag{9}$$

Thus we have

$$g_i\varphi = c_i(g_i\psi), \tag{10}$$

where $c_i \in C$ for $i = 1, ..., l$.

Then we apply a the homomorphism μ defined by the map $g_i \mapsto c_i$ for $i = 1, ..., l.$. As we noted above, $1 \sim_{\varphi,\psi} w$ if and only if the element c belongs to the image $(Eq_{C,\varphi',\psi}(G))\mu = gp(c_1, ..., c_l)$. So, our problem is reduced to the membership problem in a finitely generated abelian group C. It is well known that this problem is decidable. $\square$

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